Fundamentals of Communications Engineering

Department of Communications Engineering, College of Engineering, University of Diyala, 2016-2017

Class: Second Year
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Room: Comm-02

Lecture: 04

Convolution Integral

* Finding the effect of a signal on another signal is called convolution.

* The convolution between two signals $f_1(t)$ and $f_2(t)$,

$$f_1(t) \circledast f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \quad \text{——— (70)}$$

* Convolution integral is commutative:

$$f_1(t) \circledast f_2(t) = f_2(t) \circledast f_1(t) \quad \text{——— (71)}$$

* Convolution integral is Distributive:

$$f_1(t) \circledast \left[f_2(t) + f_3(t) \right] = f_1(t) \circledast f_2(t) + f_1(t) \circledast f_3(t) \quad \text{——— (72)}$$

* Convolution integral is Associative:

$$f_1(t) \circledast \left[f_2(t) \circledast f_3(t) \right] = \left[f_1(t) * f_2(t) \right] * f_3(t) \quad \text{——— (73)}$$

* Convolution can be shifted :

if $f_1(t) \circledast f_2(t) = g(t)$ then

$$f_1(t) \circledast f_2(t-t_0) = g(t-t_0) \quad (74)$$

$$f_1(t-t_0) \circledast f_2(t) = g(t-t_0) \quad (75)$$

$$f_1(t-t_0) \circledast f_2(t-t_1) = g(t-t_0-t_1) \quad (76)$$

* Convolution with $\delta(t)$:

$$f(t) \circledast \delta(t) = f(t) \quad (77)$$

Hence: Convolution of Any Signal with $\delta(t)$ is the Signal itself.

System Impulse Response

* In Fig.(1), the system can be Described mathematically as,

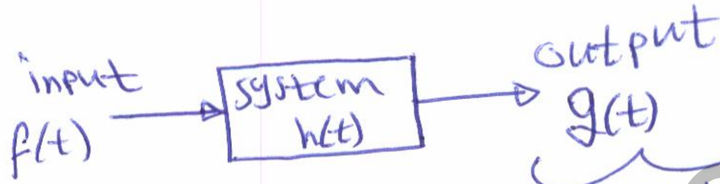


Fig.(1)

we call the output as system Response

$$\boxed{g(t) = f(t) * h(t)} \qquad (77)$$

* Thus, to get the effect of any system on a signal, all we need is to do convolution.

* In other words, Convolution integral is an operation which gives the effect of a system or signal on a system or a signal.

$\overset{1}{\text{signal}} \circledast \overset{2}{\text{signal}}$ (convolution) = effect of 1 on 2
effect of 2 on 1

signal $\circledast$ system (convolution) = effect of signal on system /or
effect of system on signal

Signal $f(t)$ $\xrightarrow{\text{input}}$ $\boxed{\text{Convolution with } h(t)}$ $\longrightarrow$ System Response $g(t)$

$\therefore$ $\boxed{\text{System response} = \text{input} \circledast \text{system}}$

* As mentioned previously :- convolution of any signal with $\delta(t)$ is the signal itself, then to get the system's function (system transfer function) the input to the unknown system $h(t)$ must be $\delta(t)$.

$\delta(t) \longrightarrow \boxed{\text{Unknown system } h(t)} \longrightarrow h(t)$

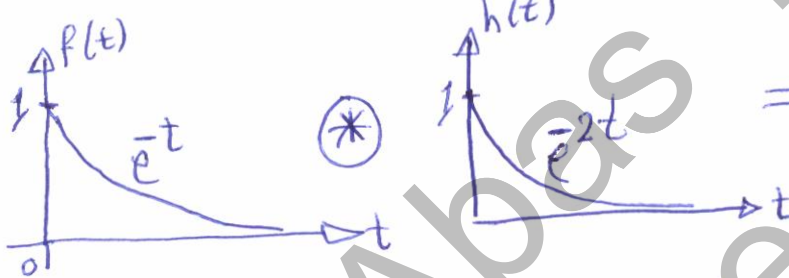
Fig. (2)

* In Fig. (2), the output $h(t)$ is called the system impulse response (which is the transfer function of the system).

EX. For a system with impulse response $h(t) = e^{-2t}u(t)$, determine the response $g(t)$ for the input $e^{-t}u(t)$.

Solution

$$e^{-t}u(t) \longrightarrow \boxed{h(t) = e^{-2t}u(t)} \longrightarrow g(t)$$



$$= g(t) = f(t) \circledast h(t)$$

* both functions starts at zero, $t \geq 0$, then

$$g(t) = f(t) \circledast h(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau \qquad t \geq 0$$

we have $f(\tau) = e^{-\tau}u(\tau)$ & $h(t-\tau) = e^{-2(t-\tau)}u(t-\tau)$

then
$$g(t) = \int_0^t e^{-\tau} e^{-2(t-\tau)}\, d\tau = \int_0^t e^{-\tau} e^{-2t} e^{2\tau}\, d\tau \qquad t \geq 0$$

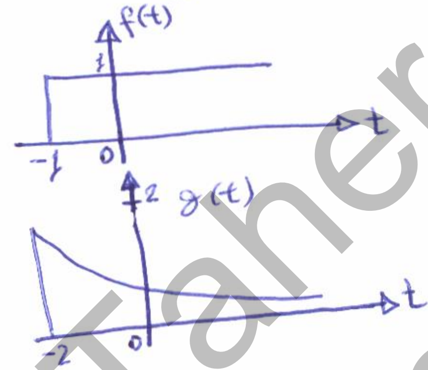
$$= e^{-2t}\int_0^t e^{\tau}\, d\tau = e^{-2t}\left[e^{\tau}\right]_0^t = e^{-2t}\left[e^{t} - 1\right] \qquad t \geq 0$$

$$= e^{-2t}e^{t} - e^{-2t} = e^{-t} - e^{-2t} \qquad t \geq 0$$

$\therefore$ $$\boxed{g(t) = \left[e^{-t} - e^{-2t}\right]u(t)}$$

Graphical Convolution

* Lets understand graphical convolution by an example,

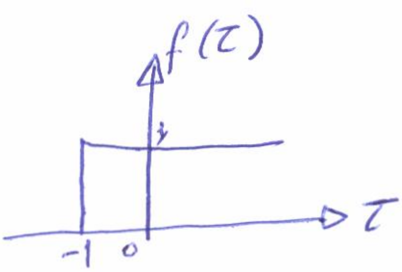


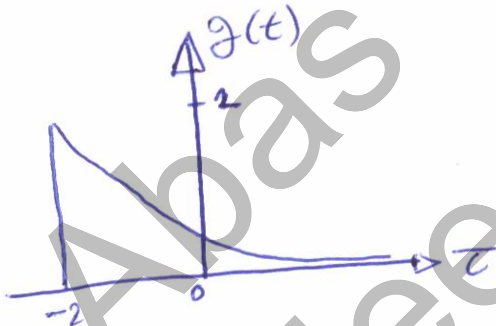
Let $f(t) = u(t+1)$

$g(t) = 2e^{-(t+2)}$

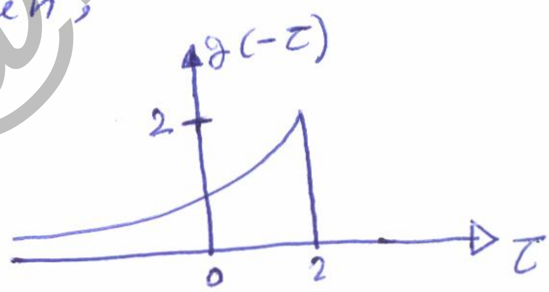
$c(t) = f(t) \circledast g(t)$

#Step 1 replace each t with τ,





$$c(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$ then,



#Step 2 plot $g(-\tau)$

#Step 3 shift $g(-\tau)$ by t as follows:

* assume $g(-\tau) = \phi(\tau)$

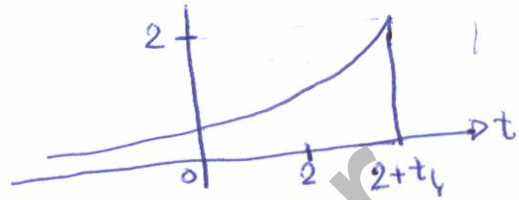
* now $\phi(\tau)$ shifted by t seconds is $\phi(\tau - t)$

$\therefore \phi(\tau - t) = g(-(\tau - t)) = g(t - \tau)$

* if $+t \longrightarrow$ shift to the right

* if $-t \longrightarrow$ Shift to the left

Thus, we start to shift to the right by t_1 to get $g(t_1-\tau)$, $t=t_1>0$



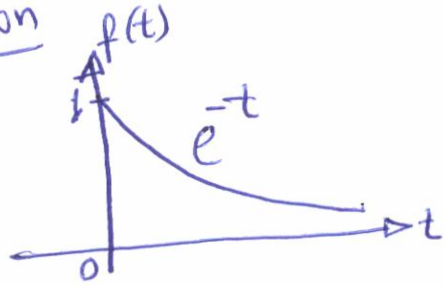
Step 4 : Move $g(\tau)$ (scanning) over $f(\tau)$.

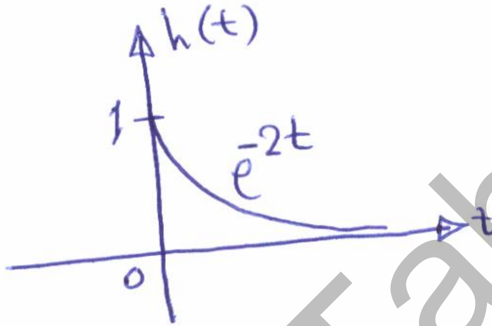
– The area under the product of $f(\tau)$ and $g(t_1-\tau)$ (the shifted frame) is $c(t_1)$, the value of the convolution at $t=t_1$.

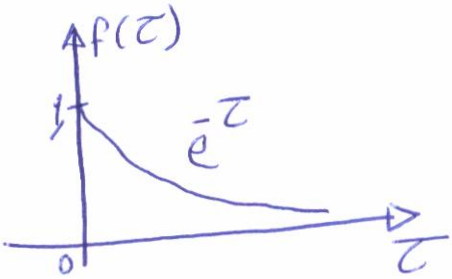
step 5 : Repeat this procedure shifting the frame by different values (positive & negative) to obtain $c(t)$ for all values of t.

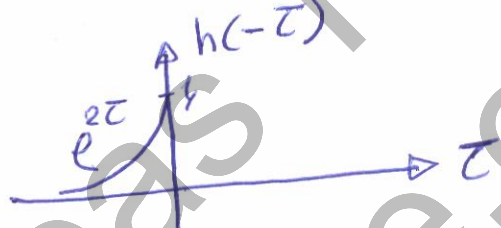
EX. Determine graphically $y(t) = f(t) \circledast h(t)$ for $f(t) = e^{-t}u(t)$ and $h(t) = e^{-2t}u(t)$.

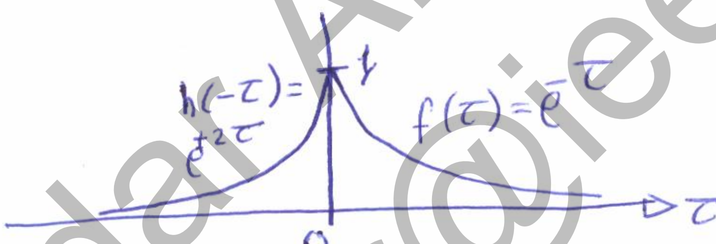
Solution



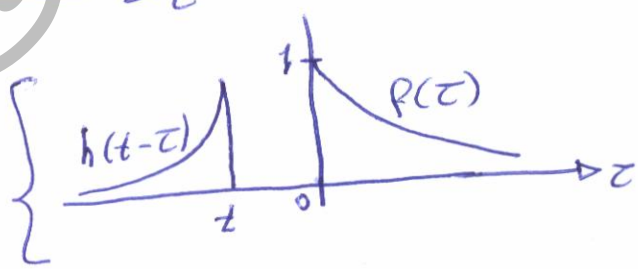








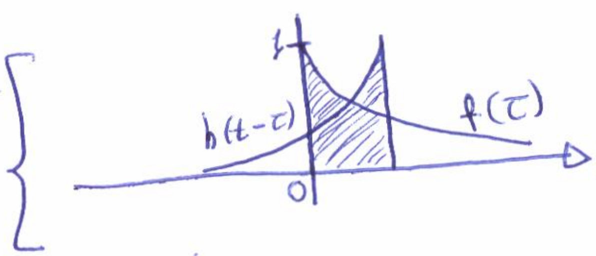
* In this case, there is no overlap:



$f(\tau)h(t-\tau) = 0$

$\therefore y(t) = 0 \quad , t < 0$

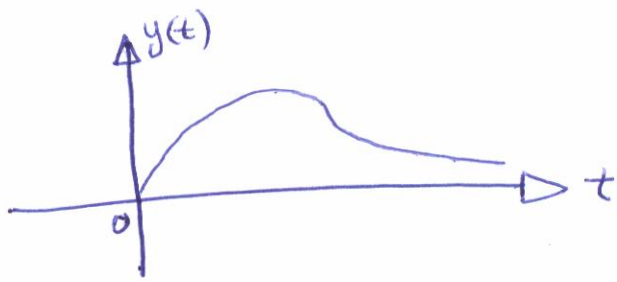
* In this case, there is overlap in the region $0 \leq \tau \leq t$ (shaded area)



$$y(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau \qquad t \geq 0$$

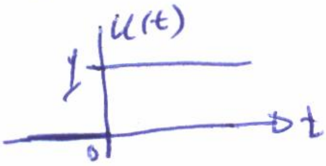
$$y(t) = \int_0^t e^{-\tau} e^{-2(t-\tau)}\, d\tau = e^{-2t} \int_0^t e^{\tau}\, d\tau = e^{-t} - e^{-2t} \qquad t \geq 0$$

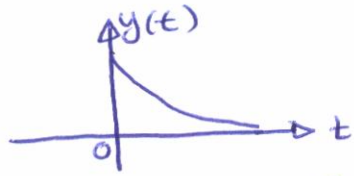
$$\therefore y(t) = \left(e^{-t} - e^{-2t}\right) u(t)$$



EX.1 convolve $x(t) = u(t)$ with $y(t) = e^{-t}u(t)$.

solution

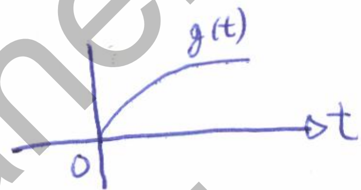




let $g(t) = u(t) \circledast e^{-t}u(t) = x(t) \circledast y(t)$

$$g(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau = \int_{0}^{t} 1\; e^{-(t-\tau)}\, d\tau$$

$$= e^{-t}\int_{0}^{t} e^{\tau}\, d\tau = e^{-t}\left[e^{t}-1\right] = 1 - e^{-t}$$



EX.2 given $r(t) = t \quad t \geq 0$ & $x(t) = e^{-\alpha t}u(t)$, find the convolution $g(t) = r(t) \circledast x(t)$.

solution

$$g(t) = \int_{-\infty}^{\infty} r(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} \tau\, e^{-\alpha(t-\tau)}\, u(t-\tau)\, d\tau$$

$$= \int_{0}^{t} \tau\, e^{-\alpha(t-\tau)}\, d\tau \qquad t \geq 0$$

$$= e^{-\alpha t}\int_{0}^{t} \tau\, e^{\alpha\tau}\, d\tau \qquad t \geq 0 \qquad \left\{\begin{array}{l}\text{integrate it using}\\ \text{by parts method}\end{array}\right\}$$

$$= \frac{t}{\alpha} - \frac{1}{\alpha^2}\left(1 - e^{-\alpha t}\right) \qquad t \geq 0$$

$$\therefore\; g(t) = \left[\frac{t}{\alpha} - \frac{1}{\alpha^2}\left(1 - e^{-\alpha t}\right)\right] u(t).$$

EX: convolve $u(t)$ with itself.

solution

$$y(t) = u(t) \circledast u(t) = \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} u(\tau)\, u(t-\tau)\, d\tau = \int_{0}^{t} d\tau = \begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$$

OR $y(t) = t\, u(t)$.

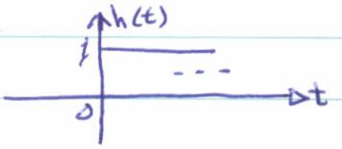
EX: find the effect of the system $h(t) = \frac{1}{t+1} u(t)$ on the input signal $x(t) = u(t)$.

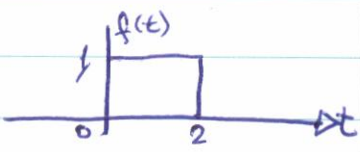
solution

$*$ input signal $x(t) = u(t)$ (input is DC)

$$g(t) = \int_{-\infty}^{\infty} u(t-\tau) \frac{1}{\tau+1}\, d\tau = \int_{0}^{t} \frac{1}{\tau+1}\, d\tau = \ln(t+1)\, u(t).$$

EX: How the system $h(t) = u(t)$ will response if its input is $f(t) = u(t) - u(t-2)$?



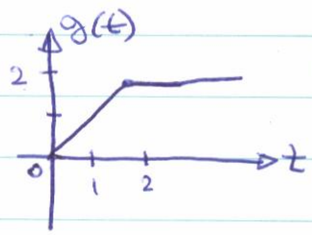


solution

$$g(t) = \int_{-\infty}^{\infty} [u(\tau) - u(\tau-2)]\, u(t-\tau)\, d\tau$$

$*$ if $\tau < 0$ and $\tau > 2 \rightarrow g(t) = 0$

$\tau > 0 \rightarrow g(t) = 0$



$*$ if $0 < t < 2 \rightarrow g(t) = \int_{0}^{t} d\tau = t$

$*$ if $t > 2 \rightarrow g(t) = \int_{0}^{2} d\tau = 2$

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2 Continuous-Time Convolution

The input, $x(t)$, and output, $y(t)$, of a continuous-time LTI system are related by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{8}$$

where $h(t)$ is the impulse response of the system. Consider a system with impulse response and input shown in Fig. 4 and given by

$$h(t) = e^{2t}u(1-t) \tag{9}$$
$$x(t) = u(t) - 2u(t-2) + u(t-5). \tag{10}$$

Substituting these into (8) yields a complicated looking expression,

$$y(t) = \int_{-\infty}^{\infty} [u(t-\tau) - 2u(t-2-\tau) + u(t-5-\tau)]\, e^{2\tau}u(1-\tau)d\tau. \tag{11}$$

This integration problem can be simplified by considering various intervals for t. Flipped and shifted versions of the impulse response are shown in Fig. 5 for these ranges of t. When $t < 1$ the curves only overlap between $t-5$ and t, see Fig. 5a, so this limits the integration interval. Furthermore, $x(t)$ is a constant value from $t-5$ to $t-2$ and a different constant value from $t-2$ to t, so separating the integral into two terms will greatly simplify our work.

$$\begin{aligned}
y(t) &= \int_{t-5}^{t} [u(t-\tau) - 2u(t-2-\tau) + u(t-5-\tau)]\, e^{2\tau}d\tau. \\
&= \int_{t-5}^{t-2} (-1)e^{2\tau}d\tau + \int_{t-2}^{t} e^{2\tau}d\tau \\
&= -\frac{1}{2}\left[e^{2(t-2)} - e^{2(t-5)}\right] + \frac{1}{2}\left[e^{2t} - e^{2(t-2)}\right] \\
y(t) &= \frac{1}{2}\left[1 - 2e^{-4} + e^{-10}\right]e^{2t} \qquad t < 1
\end{aligned} \tag{12}$$

When $t > 1 > t-2$ (i.e. $1 < t < 3$), the leading edge of $x(t-\tau)$ has shifted out beyond the end of $h(\tau)$, see Fig. 5b. Therefore, the upper limit on the integration becomes the end of $h(\tau)$.

$$\begin{aligned}
y(t) &= \int_{t-5}^{t-2} -e^{2\tau}d\tau + \int_{t-2}^{1} e^{2\tau}d\tau \\
&= -\frac{1}{2}\left[e^{2(t-2)} - e^{2(t-5)}\right] + \frac{1}{2}\left[e^{2} - e^{2(t-2)}\right] \\
y(t) &= \frac{1}{2}\left[-2e^{-4} + e^{-10}\right]e^{2t} + \frac{1}{2}e^{2} \qquad 1 < t < 3
\end{aligned} \tag{13}$$

When $t-5 < 1 < t-2$ (i.e. $3 < t < 6$), the leading section $x(t-\tau)$ has shifted out beyond the end of $h(\tau)$, so the second integral is zero and the upper limit on the first integral is the end of $h(\tau)$, see Fig. 5c.

$$y(t) = \int_{t-5}^{1} -e^{2\tau}d\tau = \frac{1}{2}\left[e^{2(t-5)} - e^{2}\right] \qquad 3 < t < 6 \tag{14}$$

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When $t-5>1$ (i.e. $t>6$), the curves no longer overlap and

$$y(t) \;=\; 0. \qquad\qquad t>6 \tag{15}$$

Putting (12) through (15) together yields the final answer,

$$y(t) \;=\; \begin{cases} \frac{1}{2}\left[1-2e^{-4}+e^{-10}\right]e^{2t} & t \le 1 \\ \frac{1}{2}\left[-2e^{-4}+e^{-10}\right]e^{2t}+\frac{1}{2}e^{2} & 1 < t \le 3 \\ \frac{1}{2}\left[e^{2(t-5)}-e^{2}\right] & 3 < t \le 6 \\ 0 & 6 < t \end{cases} \tag{16}$$

We can use unit step functions to rewrite (16) as

$$\begin{aligned} y(t) &= \tfrac{1}{2}\left[1-2e^{-4}+e^{-10}\right]e^{2t}u(1-t)+\left(\tfrac{1}{2}\left[-2e^{-4}+e^{-10}\right]e^{2t}+\tfrac{1}{2}e^{2}\right)\left[u(3-t)-u(1-t)\right] \\ &\quad +\tfrac{1}{2}\left[e^{2(t-5)}-e^{2}\right]\left[u(6-t)-u(3-t)\right] \\ &= \left(\tfrac{1}{2}\left[1-2e^{-4}+e^{-10}\right]e^{2t}-\tfrac{1}{2}\left[-2e^{-4}+e^{-10}\right]e^{2t}-\tfrac{1}{2}e^{2}\right)u(1-t) \\ &\quad +\left(\tfrac{1}{2}\left[-2e^{-4}+e^{-10}\right]e^{2t}+\tfrac{1}{2}e^{2}-\tfrac{1}{2}\left[e^{2(t-5)}-e^{2}\right]\right)u(3-t) \\ &\quad +\tfrac{1}{2}\left[e^{2(t-5)}-e^{2}\right]u(6-t) \end{aligned}$$

$$y(t) \;=\; \tfrac{1}{2}\left(e^{2t}-e^{2}\right)u(1-t)+\left(-e^{2t-4}+e^{2}\right)u(3-t)+\tfrac{1}{2}\left(e^{2t-10}-e^{2}\right)u(6-t) \tag{17}$$

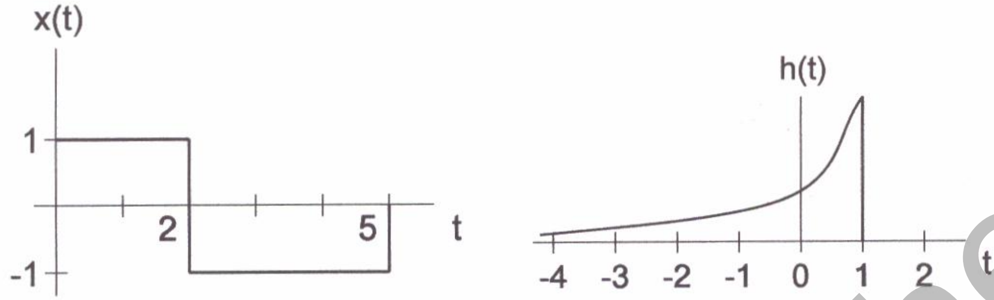
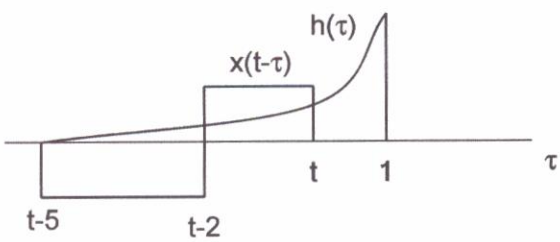
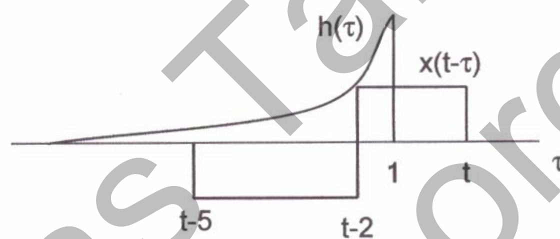


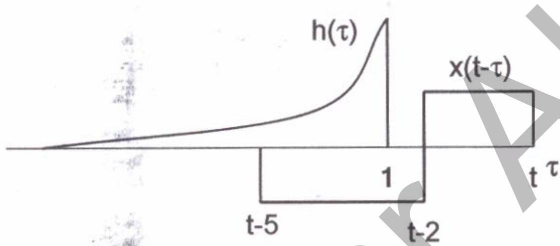
Figure 4: Input signal and system impulse response.



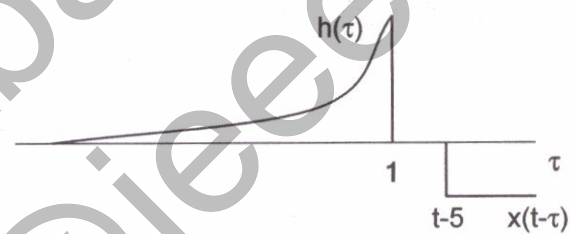
(a) Case where $t < 1$



(b) Case where $t - 2 < 1 < t$



(c) Case where $t - 5 < 1 < t - 2$



(d) Case where $1 < t - 5$

Figure 5: Overlapping curves for various values of t.

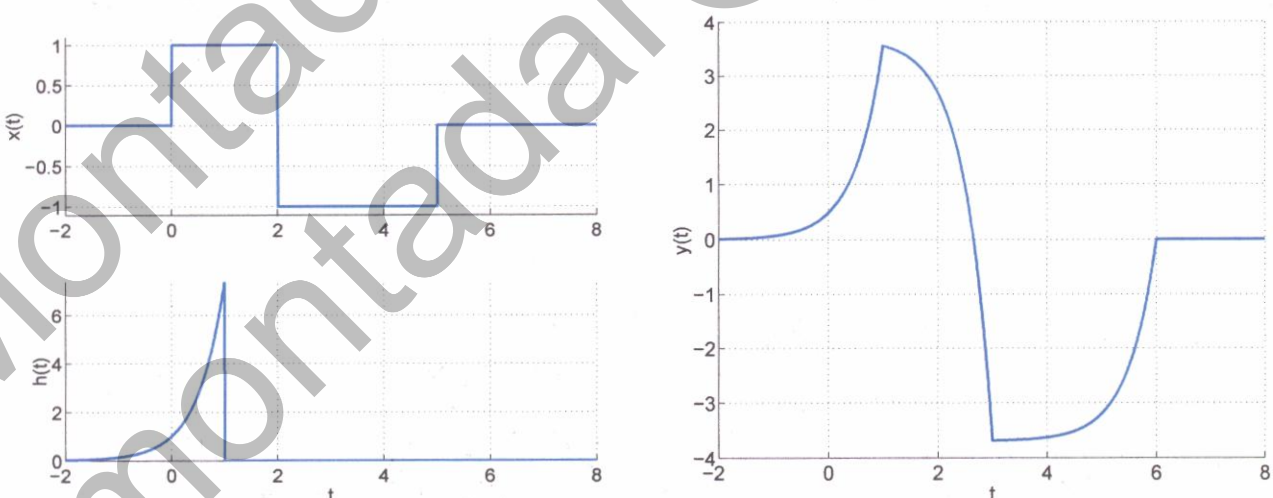


Figure 6: Continuous-time convolution example.